hep–th/0507169

Supersymmetric solutions based on $Y^{p,q}$ and $L^{p,q,r}$

Konstadinos Sfetsos and **Dimitrios Zoakos**

Department of Engineering Sciences, University of Patras
26110 Patras, Greece
sfetsos@des.upatras.gr, dzoakos@upatras.gr

Abstract

We explicitly realize supersymmetric cones based on the five-dimensional $Y^{p,q}$ and $L^{p,q,r}$ Einstein–Sasaki spaces. We use them to construct supersymmetric type-IIB supergravity solutions representing a stack of D3- and D5-branes as warped products of the six-dimensional cones and $\mathbb{R}^{1,3}$.

1 Introduction

In recent works the number of explicit examples of known five dimensional Einstein–Sasaki metrics was considerably enlarged by a new class of such metrics interpolating in a certain sense between the round S^5 sphere and the $T^{1,1}$ space [1]. These are of cohomogeneity one, with principal orbits $SU(2) \times U(1) \times U(1)$ and in order to satisfy global regularity issues their parametric space is characterized by two coprime positive integers p and q. Hence, these were called $Y^{p,q}$ spaces. This class has been further generalized by taking the BPS limits of Euclideanized Kerr–de Sitter black hole metrics with two independent angular momenta parameters [2]. This construction leads to local Einstein–Sasaki metrics of cohomogeneity two, with $U(1) \times U(1) \times U(1)$ principal orbits. Similarly, these metrics, called $L^{p,q,r}$, are characterized by positive coprime integers p, q and r in order that they smoothly extend onto complete, non-singular compact manifolds. The $Y^{p,q}$ spaces come as special limits of the $L^{p,q,r}$ ones when the angular parameters coincide and a $U(1)$ symmetry factor gets enhanced into $SU(2)$. Similarly, the $T^{1,1}$ space results by a further symmetry enhancement.

One advantage of having explicit five-dimensional regular spaces is that they can be used as a base in the construction of six-dimensional Ricci-flat cones, which in turn are basic blocks for the ten-dimensional supergravity solutions representing the gravitational field of stacks of branes and the dual description of supersymmetric gauge theories within the gauge/gravity correspondence [3]-[5]. The usual cone one constructs suffers from a singularity in its tip and therefore part of the effort is to regularize it. A basic example is the six-dimensional cone based on the $T^{1,1}$ space in which the conical singularity were first smoothened out in the so called deformed and resolved conifolds [6], by introducing a parameter, and keeping finite at the tip of the cone either an S^2 or an S^3 factor. In addition, there is also the regularized conifold in which the original curvature singularity becomes a removable bolt singularity [7]. Introducing D3-branes and taken into account their backreaction transforms the Ricci-flat solution of the cone times the Minkowski space into a warped solution of the full type-IIB supergravity [8]-[12]. We note that having a regular six-dimensional cone does not necessarily imply the regularity of the ten-dimensional solution (see, in particular, [7] that emphasizes that). The purpose of this paper is to construct the six-dimensional supersymmetric cones based on the newly discovered $Y^{p,q}$ and $L^{p,q,r}$ spaces and use them for the construction of the ten-dimensional type-IIB supergravity solutions

that include the brane backreaction.

This letter is organized as follows: In section 2 we present a brief review of the relevant aspects of the $Y^{p,q}$ and $L^{p,q,r}$ spaces. In section 3 we explicitly construct supersymmetric six-dimensional cone solutions based on these spaces. They depend on a constant moduli parameter as in the regularized conifold. In section 4 we construct supersymmetric supergravity solutions of a stack of D3- and D5-branes on these cones within type-IIB supergravity. They have the expected behaviour in the UV, but still suffer from a singularity in the IR.

2 Brief review of the $Y^{p,q}$ and $L^{p,q,r}$ spaces

In this section we provide a short review of some relevant to our construction aspects of the $Y^{p,q}$ and $L^{p,q,r}$ spaces and also comment on their relation. For details the reader should really consult the literature.

2.1 $Y^{p,q}$ geometry

The five dimensional $Y^{p,q}$ geometry in its canonical form is described by the following metric [13]-[14]

$$ds_5^2 = ds_4^2 + \left(\frac{1}{3}d\psi + \sigma\right)^2 \ , \tag{2.1}$$

where the four dimensional metric is

$$ds_4^2 = \frac{1-cy}{6}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{dy^2}{w(y)q(y)} + \frac{1}{36}w(y)q(y)(d\beta + c\cos\theta d\phi)^2 \ , \tag{2.2}$$

with

$$\sigma = -\frac{1}{3}\cos\theta d\phi + \frac{1}{3}y(d\beta + c\cos\theta d\phi) \ . \tag{2.3}$$

and

$$w(y) = \frac{2(a-y^2)}{1-cy} \ , \quad q(y) = \frac{a-3y^2+2cy^3}{a-y^2} \ . \tag{2.4}$$

Therefore, it can be seen as a $U(1)$ bundle over a four-dimensional Einstein-Kähler metric with the Kähler two-form given by $d\sigma = 2J_4$. It can be checked explicitly that the four-dimensional metric is Einstein with $R_{\mu\nu} = 6g_{\mu\nu}$ and hence the five-dimensional metric is Einstein–Sasaki with $R_{\mu\nu} = 4g_{\mu\nu}$. The coordinate y ranges between the two smallest roots

of the cubic equation $a - 3y^2 + 2cy^3 = 0$, so that the signature of the metric remains Euclidean. These are given in terms of the coprime integers p and q with explicit expressions that don't concerns us here. In order also to obtain a compact manifold the coordinate α has a finite range. The remaining ones θ, ϕ and ψ have periods π, 2π and 2π, respectively.

2.2 $L^{p,q,r}$ geometry

The five-dimensional $L^{p,q,r}$ geometry is described by the following metric [2, 15]

$$ds_5^2 = ds_4^2 + (d\tau + \sigma)^2 \ , \tag{2.5}$$

where the four-dimensional metric is

$$\begin{aligned} ds_4^2 &= \frac{\rho^2 dx^2}{4\Delta_x} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_x}{\rho^2}\left(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi\right)^2 \\ &\quad + \frac{\Delta_\theta \sin^2\theta\cos^2\theta}{\rho^2}\left[(1 - x/\alpha)d\phi - (1 - x/\beta)d\psi\right]^2 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\sigma = (1 - x/\alpha)\sin^2\theta \ d\phi + (1 - x/\beta)\cos^2\theta \ d\psi \ , \qquad \rho^2 = \Delta_\theta - x \ , \\ &\Delta_x = x(\alpha - x)(\beta - x) - \mu \ , \qquad \Delta_\theta = \alpha\cos^2\theta + \beta\sin^2\theta \ . \end{aligned} \tag{2.7}$$

The five-dimensional metric has the standard form, as in the $Y^{p,q}$ case. The parameter μ is trivial and can be set to any non-zero constant by rescaling α, β, and x, hence the metric depend on two parameters. The principal orbits $U(1) \times U(1) \times U(1)$ of the metric degenerate at $\theta = 0$ and $\theta = \frac{\pi}{2}$ and at the roots of the cubic function Δ_x. In order to obtain metrics on non-singular manifolds the ranges of the coordinates should be $0 < \theta < \frac{\pi}{2}$ and $x_1 < x < x_2$, where x_1 and x_2 are the two smallest roots of the equation $\Delta_x = 0$. The ranges of the coordinates ϕ and ψ are determined using the notion of "surface gravity", important in black hole solutions of Lorentzian signature. The analysis of the behavior at each collapsing orbit can be realized by examining the associated Killing vector ℓ whose length vanishes at the degeneration surface. By normalizing the Killing vector so that its "surface gravity" κ is equal to unity, one obtains a translation generator $\partial/\partial\chi$, where χ is a local coordinate near the degeneration surface. The metric extends smoothly onto the surface if χ has period 2π. The "surface gravity" is

$$\kappa^2 = \frac{g^{\mu\nu}(\partial_\mu \ell^2)(\partial_\nu \ell^2)}{4\ell^2} \ , \tag{2.8}$$

in the limit the degeneration surface is reached. At the degeneration surfaces $\theta = 0$ and $\theta = \frac{\pi}{2}$ the normalized killing vectors are $\partial/\partial\phi$ and $\partial/\partial\psi$ respectively, so the periodicity of the coordinates ϕ and ψ is 0 to 2π. At the degeneration surfaces $x = x_1$ and $x = x_2$, the associated normalized Killing vectors ℓ_1 and ℓ_2 are given by

$$\ell_i = c_i \frac{\partial}{\partial \tau} + a_i \frac{\partial}{\partial \phi} + b_i \frac{\partial}{\partial \psi} \,, \tag{2.9}$$

where the constants c_i, a_i and b_i are given by

$$a_i = \frac{\alpha c_i}{x_i - \alpha} \,, \qquad b_i = \frac{\beta c_i}{x_i - \beta} \,, \qquad c_i = \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha+\beta)x_i - \alpha\beta - 3x_i^2} \,. \tag{2.10}$$

Similarly to the case of $Y^{p,q}$, all parameters are eventually given in terms of three coprime positive integers p, q and r so that the manifolds are complete and free of singularities.

2.2.1 Connection with $Y^{p,q}$

If one sets $p + q = 2r$, implying $\alpha = \beta$, the metric (2.5) reduce to (2.2) with $Y^{p,q} = L^{p-q,p+q,p}$. Then the relation of variables and parameters is given by [16]

$$x \to \frac{\alpha}{3}(1 + 2cy) \,, \quad \theta \to \frac{\theta}{2}, \quad \phi - \psi \to -\phi \,, \quad \phi + \psi \to \frac{\beta}{c} \,, \quad 3\tau + \phi + \psi \to -\psi \tag{2.11}$$

and

$$\mu = \frac{4}{27}(1 - ac^2)\alpha^3. \tag{2.12}$$

After the coordinate transformation (2.11) the Killing vectors for the degeneration surfaces $\theta = 0$ and $\theta = \pi$ are $(-\partial/\partial\phi - \partial/\partial\psi + c\partial/\partial\beta)$ and $(\partial/\partial\phi - \partial/\partial\psi + c\partial/\partial\beta)$, respectively. At the degeneration surfaces $y = y_1$ and $y = y_2$, where y_1 and y_2 are the roots of the equation $q(y) = 0$, the normalized killing vectors ℓ_1 and ℓ_2 are given by

$$\ell_i = \frac{\partial}{\partial \psi} - \frac{1}{y_i}\frac{\partial}{\partial \beta} \,, \qquad i = 1, 2 \tag{2.13}$$

and correspond to the vectors in (2.9).

3 The six-dimensional cones

In this section we explicitly solve the supersymmetric Killing spinor equations and determine the six-dimensional cones. The latter are by construction Ricci-flat.

3.1 The cone over the $Y^{p,q}$ space

First we construct a six-dimensional supersymmetric cone over the $Y_{p,q}$ space as a base. The metric ansatz is

$$ds_6^2 \;=\; dr^2 + A(r)^2\left(\frac{1}{3}d\psi + \sigma\right)^2 + B(r)^2 ds_4^2 \ . \tag{3.1}$$

We will use the vielbein basis

$$\begin{aligned}
&e^1 = B(r)\sqrt{\frac{1-cy}{6}}d\theta \ , \qquad e^2 = B(r)\sqrt{\frac{1-cy}{6}}\sin\theta d\phi \ , \\
&e^3 = B(r)\frac{dy}{\sqrt{w(y)q(y)}} \ , \qquad e^4 = B(r)\frac{1}{6}\sqrt{w(y)q(y)}(d\beta + c\cos\theta d\phi) \ , \\
&e^5 = A(r)\frac{1}{3}\left[d\psi - \cos\theta d\phi + y(d\beta + c\cos\theta d\phi)\right] \ , \qquad e^6 = dr \ .
\end{aligned} \tag{3.2}$$

The non-vanishing components of the spin connection are

$$\begin{aligned}
&\omega^{12} = -\frac{1}{B}\left[\cot\theta\left(\frac{6}{1-cy}\right)^{1/2} e^2 + \frac{A}{B}e^5 - \frac{c}{2(1-cy)}\sqrt{wq}\; e^4\right] \ , \\
&\omega^{34} = -\frac{1}{B}\left[\frac{\partial}{\partial y}\sqrt{wq}\; e^4 + \frac{A}{B}e^5\right] \ , \\
&\omega^{14} = \frac{1}{B}\frac{c}{2(1-cy)}\sqrt{wq}\; e^2 \ , \qquad \omega^{15} = -\frac{A}{B^2}e^2 \ , \\
&\omega^{13} = -\frac{1}{B}\frac{c}{2(1-cy)}\sqrt{wq}\; e^1 \ , \qquad \omega^{25} = \frac{A}{B}e^1 \ , \\
&\omega^{24} = -\frac{1}{B}\frac{c}{2(1-cy)}\sqrt{wq}\; e^1 \ , \qquad \omega^{45} = \frac{A}{B^2}e^3 \ , \\
&\omega^{23} = -\frac{1}{B}\frac{c}{2(1-cy)}\sqrt{wq}\; e^2 \ , \qquad \omega^{35} = -\frac{A}{B^2}e^4 \ , \\
&\omega^{i6} = \frac{B'}{B}e^i, \quad i = 1\ldots 4 \ , \qquad \omega^{56} = \frac{A'}{A}e^5 \ ,
\end{aligned} \tag{3.3}$$

where prime denotes differentiation with respect to r.[1] The Killing spinor equation are

$$\partial_\mu \epsilon + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon = 0 \ . \tag{3.4}$$

[1] One might try a more general ansatz than (3.1) by putting different functions of r in front of every vielbein. However, it turns that the consistent with supersymmetry solution in the end simplifies the ansatz to that in (3.1). This is consistent with the observation of [17] that, generically the $Y^{p,q}$ manifolds do not admit complex deformations.

In analyzing this set of equations we found necessary to impose the two projections

$$\Gamma_{12}\epsilon = \Gamma_{34}\epsilon = -\Gamma_{56}\epsilon \ , \tag{3.5}$$

hence reducing supersymmetry to 1/4 of the maximal. The Killing spinor turns out to be

$$\epsilon = e^{\frac{1}{2}\psi\Gamma_{12}}\epsilon_0 \ . \tag{3.6}$$

In addition we obtained the following system of differential equations that determine the functions $A(r)$ and $B(r)$

$$B' = \frac{A}{B} \ , \qquad A' = 3 - 2\frac{A^2}{B^2} \ . \tag{3.7}$$

The general solution to the system is

$$B^2 = R^2 \ , \qquad A^2 = R^2\left(1 + \frac{C}{R^6}\right) \ , \tag{3.8}$$

where C is a constant. The relation of the two variables r and R is via the differential

$$dr = \left(1 + \frac{C}{R^6}\right)^{-1/2} dR \ . \tag{3.9}$$

Note that we have absorbed a second integration constant by a suitable redefinition of the variable R. After substituting the solution of the killing spinor equations to (3.1) the metric takes the simple form[2]

$$ds_6^2 = \left(1 + \frac{C}{R^6}\right)^{-1} dR^2 + R^2\left(1 + \frac{C}{R^6}\right)\left(\frac{1}{3}d\psi + \sigma\right)^2 + R^2 ds_4^2 \ . \tag{3.10}$$

We have checked that this metric has the same killing vectors with (2.2), with degeneration surfaces $\theta = 0$, $\theta = \pi$, $y = y_1$ and $y = y_2$.

The asymptotic behavior for large values of R takes the universal form

$$ds_6^2 \simeq dR^2 + R^2 ds_5^2, \quad \text{as} \quad R \to \infty \tag{3.11}$$

and it describes the usual cone whose base is given by the five dimensional metric (2.2). This solution is exact for all values of R since it can be obtained by simply letting $C = 0$. The constant C changes the solution drastically towards the interior. When $C \geq 0$, the variable $R \geq 0$ and then the manifold has a curvature singularity at $R = 0$. However, if

[2] This solution belongs to the class of examples considered in [18, 19] by solving the second order field equations. We thank C. Pope for the information. A form of this solution was also obtained in [20] but without any claim or proof on supersymmetry.

$C=-a^6<0$, where a is a real positive constant, then the variable $R\geq a$. To examine the behaviour of the metric near $R=a$ we change into the new radial variable $t=\sqrt{6a(R-a)}$. We find

$$ds_6^2\simeq a^2ds_4^2+\frac{1}{9}dt^2+t^2\left(\frac{1}{3}d\psi+\sigma\right)^2\quad \text{as}\quad t\to 0\ . \tag{3.12}$$

Therefore, near $t=0$ and for constant y, θ, β and ϕ, the metric behaves (up to 1/9) as $dt^2+t^2d\psi^2$ which shows that $t=0$ is a *bolt* singularity [21] which is removable since the periodicity of the angle is $0\leq\psi<2\pi$. The full solution interpolates between (3.12) for $R\to a$ and (3.11) for $R\to\infty$. This is similar to that found in [22] for the cones over the symmetric coset spaces $SU(2)^n/U(1)^{n-1}$ that includes the regularization of the singular conifold on $T^{1,1}$ for $n=2$ [7]. However, in our case we do not have a completely non-singular solution at the supergvavity level.[3] The Einstein–Kahler four-dimensional base is singular. At best it has orbifold singularities, when $4p^2-3q^2=n^2$, where $n\in Z$. The $Y^{p,q}$ metrics are then an orbifold $U(1)$ bundle over this Einstein–Kahler base orbifold [14]. Nevertheless, string theory has probably more success with orbifold singularities than true curvature singularities since in some cases the singularity is resolved before the "smoothening" [23]. It is interesting to investigate this further.

3.2 The cone over the $L^{p,q,r}$ space

To construct the six-dimensional supersymmetric cone over $L_{p,q,r}$ we make the ansatz

$$ds_6^2\ =\ dr^2+A(r)^2\left(d\tau+\sigma\right)^2+B(r)^2ds_4^2 \tag{3.13}$$

and use the vielbein basis

$$\begin{aligned}
&e^1=B(r)\frac{\rho}{\Delta_\theta^{1/2}}d\theta\ ,\qquad e^2=B(r)\frac{\Delta_\theta^{1/2}\sin\theta\cos\theta}{\rho}\left(\frac{\alpha-x}{\alpha}d\phi-\frac{\beta-x}{\beta}d\psi\right)\ ,\\
&e^3=B(r)\frac{\Delta_x^{1/2}}{\rho}\left(\frac{\sin^2\theta}{\alpha}d\phi+\frac{\cos^2\theta}{\beta}d\psi\right)\ ,\qquad e^4=B(r)\frac{\rho}{2\Delta_x^{1/2}}dx\\
&e^5=A(r)\left(d\tau+\sigma\right)\ ,\qquad e^6=dr\ .
\end{aligned} \tag{3.14}$$

After some tedious algebra we found that the non-vanishing components of the spin connection are

$$\omega^{12}=-\frac{1}{B}\left[\left(2\cot 2\theta\frac{\Delta_\theta^{1/2}}{\rho}-\frac{\alpha-\beta}{2}\sin 2\theta\left(\frac{1}{\rho\Delta_\theta^{1/2}}-\frac{\Delta_\theta^{1/2}}{\rho^3}\right)\right)e^2+\frac{\Delta_x^{1/2}}{\rho^3}e^3+\frac{A}{B}e^5\right]\ ,$$

[3] We thank C. Pope for a correspondence on this.

$$\omega^{34} = \frac{1}{B}\left[\left(\frac{3x^2 - 2(\alpha+\beta)x + \alpha\beta}{\rho\Delta_x^{1/2}} + \frac{\Delta_x^{1/2}}{\rho^3}\right) e^3 - \frac{A}{B}e^5 + \frac{\alpha-\beta}{2}\sin 2\theta \frac{\Delta_\theta^{1/2}}{\rho^3} e^2\right] ,$$

$$\omega^{14} = -\frac{1}{B}\left[\frac{\Delta_x^{1/2}}{\rho^3}e^1 - \frac{\alpha-\beta}{2}\sin 2\theta\frac{\Delta_\theta^{1/2}}{\rho^3}e^4\right] , \qquad \omega^{15} = -\frac{A}{B^2}e^2 ,$$

$$\omega^{13} = -\frac{1}{B}\left[\frac{\Delta_x^{1/2}}{\rho^3}e^2 - \frac{\alpha-\beta}{2}\sin 2\theta\frac{\Delta_\theta^{1/2}}{\rho^3}e^3\right] , \qquad \omega^{25} = \frac{A}{B^2}e^1 , \tag{3.15}$$

$$\omega^{24} = -\frac{1}{B}\left[\frac{\Delta_x^{1/2}}{\rho^3}e^2 - \frac{\alpha-\beta}{2}\sin 2\theta\frac{\Delta_\theta^{1/2}}{\rho^3}e^3\right], \qquad \omega^{45} = \frac{A}{B^2}e^3 ,$$

$$\omega^{23} = +\frac{1}{B}\left[\frac{\Delta_x^{1/2}}{\rho^3}e^1 - \frac{\alpha-\beta}{2}\sin 2\theta\frac{\Delta_\theta^{1/2}}{\rho^3}e^4\right] , \qquad \omega^{35} = -\frac{A}{B^2}e^4 ,$$

$$\omega^{i6} = \frac{B'}{B}e^i , \qquad i = 1\ldots 4 , \qquad \omega^{56} = \frac{A'}{A}e^5 .$$

The set of projections obtained by analyzing the Killing spinor equations are the same as in the case of the cone over the $Y^{p,q}$ space in (3.5) and similarly the system of differential equations (3.7) determining the functions $A(r)$ and $B(r)$. The Killing spinor is

$$\epsilon = e^{\frac{1}{2}(3\tau+\phi+\psi)\Gamma_{12}}\epsilon_0 . \tag{3.16}$$

Finally the solution takes the simple form

$$ds_6^2 = \left(1+\frac{C}{R^6}\right)^{-1} dR^2 + R^2 ds_4^2 + R^2\left(1+\frac{C}{R^6}\right)(d\tau+\sigma)^2 . \tag{3.17}$$

The metric above has the same killing vectors with (2.5), with degeneration surfaces $\theta = 0$, $\theta = \pi/2$, $x = x_1$ and $x = x_2$. As in the case of the cone over $Y^{p,q}$, for $C = -a^6$ the metric (3.17) is free of curvature singularities, but it has the singularities associated with the four-dimensional Einstein–Kahler space.

4 Warped type-IIB solutions

In order to construct ten-dimensional supersymmetric warped solutions we utilize the procedure developed in [24, 25]. We will use the cone over the $L^{p,q,r}$ space.[4] This procedure was also recently used to construct a solution with the usual cone over $Y^{p,q}$ [27]. The first

[4]For related work with the usual cone over $L^{p,q,r}$ see also [15] and [26].

step is to find a harmonic (2,1) form $\Omega_{2,1}$. For this reason we will use the local Kähler form J_4 on the Kähler-Einstein base

$$
\begin{aligned}
J_4 &= \tilde{e}^1 \wedge \tilde{e}^2 + \tilde{e}^3 \wedge \tilde{e}^4 \\
&= \sin\theta\cos\theta d\theta \wedge [(1-x/\alpha)d\phi - (1-x/\beta)d\psi] \\
&\quad -\frac{1}{2}dx \wedge \left(1/\alpha \sin^2\theta \; d\phi + 1/\beta\cos^2\theta \; d\psi\right) \ ,
\end{aligned} \tag{4.1}
$$

where $\tilde{e}^i = e^i/A(r)$. In turn, based on [28], it is possible to construct a $\Omega_{2,1}$ form from a (1,1) form ω such that $*_4\omega = -\omega$, $d\omega = 0$ and $\omega \wedge J_4 = 0$. Such a form is similar to the one proposed in [27] and [15] for the case of the usual cone on the $Y^{p,q}$ and $L^{p,q,r}$, respectively. We have explicitly that

$$
\begin{aligned}
\omega &= \frac{1}{\rho^4}(\tilde{e}^1 \wedge \tilde{e}^2 - \tilde{e}^3 \wedge \tilde{e}^4) \\
&= \frac{1}{\rho^4}\Big[\sin\theta\cos\theta \; d\theta \wedge ((1-x/\alpha)d\phi - (1-x/\beta)d\psi) \\
&\quad +\frac{1}{2}dx \wedge (1/\alpha\sin^2\theta \; d\phi + 1/\beta\cos^2\theta \; d\psi)\Big] \ ,
\end{aligned} \tag{4.2}
$$

where the overall factor in the first line has been fixed by demanding that $d\omega = 0$.[5] In order to check that the above form is indeed $(1,1)$ we introduce the set of complex coordinates (This should be equivalent to that presented in [15] in a different coordinate system)

$$
\begin{aligned}
\eta_1 &= -\frac{\cot\theta}{\Delta_\theta}d\theta + \frac{\beta - x}{2\Delta_x}dx + \frac{i}{\alpha}d\phi \ , \\
\eta_2 &= \frac{\tan\theta}{\Delta_\theta}d\theta + \frac{\alpha - x}{2\Delta_x}dx + \frac{i}{\beta}d\psi \ , \\
\eta_3 &= \left(1 - \frac{a^6}{R^6}\right)^{-1}\frac{dR}{R} + i\tilde{e}^5 - \eta_1(\alpha - x)\sin^2\theta - \eta_2(\beta - x)\cos^2\theta \ .
\end{aligned} \tag{4.4}
$$

It can be shown that the η_i's indeed are closed and by construction $(1,0)$ forms. Using (4.4) we can solve for $d\theta, dx, d\phi$ and $d\psi$ in terms of $\eta_{1,2}$ and their complex conjugates. Then after substituting into (4.2) (and (4.3) for that matter) and some algebra we may show that both expressions indeed represent (1,1) forms.

[5] A second possibility is

$$
\omega = \frac{1}{\sin 2\theta(\Delta_\theta\Delta_x)^{1/2}}(\tilde{e}^1 \wedge \tilde{e}^4 - \tilde{e}^2 \wedge \tilde{e}^3) = -\frac{1}{2\alpha\beta}\; d\phi \wedge d\psi + \frac{\rho^2}{2\sin 2\theta\Delta_\theta\Delta_x}\; d\theta \wedge dx \ . \tag{4.3}
$$

However, this form is singular at $\theta = 0, \pi$ and $x = x_1, x_2$ and cannot be used to construct a complex 3-form with well defined associated charges. We thank C. Herzog for a correspondence on this.

Next we construct a (2,1) form as the wedge product of a (1,0) form and ω

$$\Omega_{2,1} = K \left[\left(1 - \frac{a^6}{R^6} \right)^{-1} \frac{dR}{R} + i\tilde{e}^5 \right] \wedge \omega \ , \tag{4.5}$$

where K is a normalization constant. It is easily verified that the $\Omega_{2,1}$ form is closed and imaginary self-dual in the six dimensional space, namely

$$d\Omega_{2,1} = 0 \ , \qquad *_6\Omega_{2,1} = i\Omega_{2,1} \ . \tag{4.6}$$

For the supergravity solution, we take the real RR F_3 and NSNS H_3 forms to be

$$iM\Omega_{2,1} = F_3 + \frac{i}{g_s} H_3 \tag{4.7}$$

and therefore

$$F_3 = -MK\tilde{e}^5 \wedge \omega \ , \qquad H_3 = g_s MK \left(1 + \frac{C}{R^6} \right)^{-1} \frac{dR}{R} \wedge \omega \ , \tag{4.8}$$

where M is another normalization constant. The ansatz for the warped metric of the ten-dimensional type-IIB solution is

$$ds_{10}^2 = H^{-1/2} ds_4^2 + H^{1/2} \left[\left(1 - \frac{a^6}{R^6} \right)^{-1} dR^2 + R^2 ds_4^2 + R^2 \left(1 - \frac{a^6}{R^6} \right) (d\tau + \sigma)^2 \right] \ , \tag{4.9}$$

where the warp factor H in generally depends on R, x and θ. There is no dilaton or axion field, while the self-dual five form is

$$g_s F_5 = d(H^{-1}) \wedge d^4x + *_{10} \left[d(H^{-1}) \wedge d^4x \right] \ , \tag{4.10}$$

which after some algebra takes the form

$$\begin{aligned} g_s F_5 \ &= \ -H^{-2} \left(\frac{\partial H}{\partial R} dR + \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial \theta} d\theta \right) \wedge d^4x \\ &\quad - \frac{\partial H}{\partial R} R^5 \left(1 - \frac{a^6}{R^6} \right) \frac{\sin 2\theta}{4\alpha\beta} \rho^2 d\tau \wedge d\theta \wedge dx \wedge d\phi \wedge d\psi \\ &\quad - \frac{\partial H}{\partial x} R^3 \frac{\sin 2\theta}{\alpha\beta} \Delta_x d\tau \wedge dR \wedge d\theta \wedge d\phi \wedge d\psi \\ &\quad + \frac{\partial H}{\partial \theta} R^3 \frac{\sin 2\theta}{4\alpha\beta} \Delta_\theta d\tau \wedge dR \wedge dx \wedge d\phi \wedge d\psi \ . \end{aligned} \tag{4.11}$$

To determine the warped factor we substitute in the Bianchi identity

$$dF_5 = H_3 \wedge F_3 \tag{4.12}$$

and obtain a second order partial differential equation whose precise form depends on which one of (4.2) or (4.3) we use to construct the 3-forms H_3 and F_3. If we use (4.2) in the Bianchi identity we obtain

$$\begin{aligned} &\frac{1}{R^3}\frac{\partial}{\partial R}\left(\frac{\partial H}{\partial R}R^5\left(1-a^6/R^6\right)\right)+\frac{4}{\rho^2}\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\Delta_x\right) \\ &+\frac{1/\rho^2}{\sin 2\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial H}{\partial\theta}\sin 2\theta\Delta_\theta\right)=-2\frac{g_s^2M^2K^2}{\rho^8R^4}\left(1-a^6/R^6\right)^{-1}\ . \end{aligned} \tag{4.13}$$

In the special case with $\alpha=\beta$ we can check that this equations indeed reduces to that in [27] after we also make the consistent assumption that H is θ-independent. We were not able to find exact solutions of (4.13) in the generic case. In that respect note that it is not consistent to assume θ-independence of the solutions. Perhaps the work of [29] who study the Laplacian in the $Y^{p,q}$ spaces will be useful in that direction as well. Nevertheless, we may easily see that for large R it exhibits the generic behaviour as $H\sim\ln R/R^4$. Towards the infrared for $R\to a$ it is seen that there is a singularity since $H\sim\ln^2(R-a)$.

Perhaps the most important open issue concerns the construction of a supergravity solution utilizing the $Y^{p,q}$ and $L^{p,q,r}$ spaces and being dual to $\mathcal{N}=1$ gauge theories, in which the IR singularity is smoothened out. Let's recall that some times a useful approach in constructing supersymmetric spaces representing cones with smoothened out singularities is via gauged supergravities. In particular, many such solutions having an $SU(2)$ isometry were found using the eight-dimensional supergravity of [30] resulting from dimensionally reducing the eleven-dimensional supergravity of [31] (see in particular the works [32]-[39][22]). The use of the lower dimensional gauged supergravity disentangles certain technical issues which are due to the complexity of the base manifolds (in our case see the expressions for the spin connections in section 3). We believe that at least for the case of solutions having the $Y^{p,q}$ manifold as an internal part the use of gauged supergravity could be proven quite useful.

Acknowledgments

We would like to thank C.N.Pope and D. Martelli for a correspondence and A. Paredes for a discussion. We acknowledge the financial support provided through the European Community's program "Fundamental Forces and Symmetries of the Universe" with contract MRTN-CT-2004-005104, the INTAS contract 03-51-6346 "Strings, branes and higher-spin gauge fields", as well as by the Greek Ministry of Education program ΠΥΘΑΓΟΡΑΣ with contract 89194. In addition D.Z. acknowledges the financial support provided through the Research Committee of the University of Patras for a "K.Karatheodory" fellowship under contract number 3022. We also thank Ecole Polytechnique and CERN for hospitality and financial support during part of this work.

References

[1] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, ATMP **8** (2004) 711, hep-th/0403002.

[2] M. Cvetic, H. Lu, D. N. Page and C. N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, hep-th/0504225 and *New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter*, hep-th/0505223.

[3] J.M. Maldacena, ATMP **2** (1998) 231, hep-th/9711200.

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. **B428** (1998) 105, hep-th/9802109.

[5] E. Witten, ATMP **2** (1998) 253, hep-th/9802150.

[6] P. Candelas and X.C. de la Ossa, Nucl. Phys. **B342** (1990) 246.
L.A. Pando Zayas and A.A. Tseytlin, JHEP **0011** (2000) 028, hep-th/0010088.

[7] L.A. Pando Zayas and A. A. Tseytlin, Phys. Rev. **D63** (2001) 086006, hep-th/0101043.

[8] A. Kehagias, Phys. Lett. **B435** (1998) 337, hep-th/9805131.

[9] I.R. Klebanov and E. Witten, Nucl. Phys. **B536** (1998) 199, hep-th/9807080.

[10] I.R. Klebanov and N.A. Nekrasov, Nucl. Phys. **B574** (2000) 263, hep-th/9911096.

[11] I.R. Klebanov and A.A. Tseytlin, Nucl. Phys. **B578** (2000) 123, hep-th/0002159.

[12] I.R. Klebanov and M.J. Strassler, JHEP **0008** (2000) 052, hep-th/0007191.

[13] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Class. Quant. Grav. **21**, 4335 (2004) hep-th/0402153.

[14] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, hep-th/0411238.

[15] D. Martelli and J. Sparks, *Toric Sasaki-Einstein metrics on* $S^2 \times S^3$, hep-th/0505027.

[16] A. Butti, D. Forcella and A. Zaffaroni, *The dual superconformal theory for* $L^{p,q,r}$ *manifolds*, hep-th/0505220.

[17] S. Franco, A. Hanany and A.M. Uranga, *Multi-flux warped throats and cascading gauge theories*, hep-th/0502113.

[18] L. Berard-Bergery, C.R. Acad. Sci, Paris, Ser. 1302 (1986) 159.

[19] D.N. Page and C.N. Pope, Class. Quant. Grav. **4** (1987) 213.

[20] S.S. Pal, Phys. Lett. **B614** (2005) 201, hep-th/0501012.

[21] G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. **66** (1979) 291.
T. Eguchi and A.J. Hanson, Annals Phys. **120** (1979) 82.

[22] R. Hernandez and K. Sfetsos, JHEP **0207** (2002) 045, hep-th/0205099 and Phys. Lett. **B582** (2004) 102, hep-th/0309147.

[23] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B261** (1985) 678 and Nucl. Phys. **B274** (1986) 285.

[24] M. Grana and J. Polchinski, Phys. Rev. **D63** (2001) 026001, hep-th/0009211.

[25] S.S. Gubser, *Supersymmetry and F-theory realization of the deformed conifold with three-form flux*, hep-th/0010010.

[26] D. Gepner and S. Pal, *Branes in* $L^{p,q,r}$, hep-th/0505039.

[27] C. P. Herzog, Q.J. Ejaz and I.R. Klebanov, JHEP **0502** (2005) 009, hep-th/0412193.

[28] S. Franco, Y. H. He, C. Herzog and J. Walcher, Phys. Rev. **D70**, 046006 (2004) hep-th/0402120 and *Chaotic cascades for D-branes on singularities*, hep-th/0412207.

[29] H. Kihara, M. Sakaguchi and Y. Yasui, *Scalar Laplacian on Sasaki-Einstein manifolds* $Y^{p,q}$, hep-th/0505259.

[30] A. Salam and E. Sezgin, Nucl. Phys. **B258** (1985), 284.

[31] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. **B76** (1978) 409.

[32] J.D. Edelstein and C. Núñez, JHEP **0104** (2001) 028, hep-th/0103167.

[33] J.D. Edelstein, A. Paredes and A.V. Ramallo, Phys. Lett. **B554** (2003) 197, hep-th/0212139.

[34] J. Gomis and T. Mateos, Phys. Lett. **B524** (2002) 170, hep-th/0108080.

[35] R. Hernández and K. Sfetsos, Phys. Lett. **B536** (2002) 294, `hep-th/0202135`.

[36] J.D. Edelstein, A. Paredes and A.V. Ramallo, JHEP **0301** (2003) 011, `hep-th/0211203`.

[37] R. Hernández, Phys. Lett. **B521** (2001) 371, `hep-th/0106055`.

[38] U. Gursoy, C. Núñez and M. Schvellinger, JHEP **0206** (2002) 015, `hep-th/0203124`.

[39] J. Brugués, J. Gomis, T. Mateos and T. Ramírez, JHEP **0210** (2002) 016, `hep-th/0207091`.